Distribution function properties and the fundamental diagram in kinetic traffic flow theory

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Traffic flow models recently discussed in the literature are mostly centered on either cellular automata or the hydrodynamic analog. The present paper reconsiders the approach of Prigogine and Herman [*Kinetic Theory of Vehicular Traffic* (American Elsevier, New York, 1971)], which hinges on the distribution function $f(x,v,t)$. While the work of Prigogine and Herman is an *ab initio* treatment, this paper presents an empirical alternative analysis regarding the fundamental diagram as an input quantity. With a series ansatz, solutions for f in both, the stationary and time-dependent case, can be obtained. A number of traffic phenomena are shown to be reproduced. [S1063-651X(96)02612-8]

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I. INTRODUCTION

Current research in vehicular traffic phenomena is conducted within several rather well-established approaches. Macroscopic model building [1,2] (going back to the landmark paper by Lighthill and Whitham [3]), on the one hand, mainly employs the analogies between certain averaged properties of traffic flow and hydrodynamics. This approach necessarily disregards details of the microscopic processes of driving that enter only in averaged form. Although, e.g., the hydrodynamical laminar-turbulent transition successfully describes jams unexpectedly on an infinitely extended single lane [1], it has to be cautioned that the driver-car unit differs drastically from a particle in a liquid. Energy and momentum conservation are not maintained, the driving forces remain unclear, and crossroads and traffic rules cannot be translated into hydrodynamic language at all. Also, the notion of a ''flow'' generated by the motion of individual vehicles is generally problematic because of the coarse-grained structure of the problem. Furthermore, the so-called fundamental diagram $\overline{v}=V(c)$, the relation between mean velocity $\overline{v}$ and concentration c, has to be given as input, whereas in an *ab initio* theory it would have to be the consequence of microscopic interactions.

Microscopic modeling, on the other hand, generally starts from car-following dynamics where the individual motion is essentially a reaction to the behavior of the car in front with kinematic restraints such as engine power, delay times, and traffic rules. Dynamics of this type may be expressed in an oversimplified yet computer-friendly manner via cellular automata [4–6] or in more detail to achieve realistic results [7]. Models of such a structure have been successfully applied to jam phenomena and the explanation of the fundamental diagram. In any case, however, microscopical simulations soon reach computational limits, which imposes restrictions on the models: Either they have to be kept too simple for any real-world significance or closing relations are introduced that are open to ambiguity. Furthermore, car-following dynamics take into account only pair interactions (i.e., with the car in front) and systematically ignore collective effects such as, e.g., the ''pressure'' exerted by the local density of traffic.

Following the landmark book by Prigogine and Herman [8], this paper will be based on the observation that the distribution function $f(x,v,t)$, giving the number of cars at time t in the phase-space interval $[\{x,v\},\{x+dx,v+dv\}]$, can act as a bridge between macroscopic and microscopic theories such as, for instance, in conventional kinetic gas theory.

Starting on the microscopic side, f can be thought of as the result of simple counting. If a set of rules is given, the change of population in a stretch of road $[x,x+L]$ within a certain time interval determines f. If then the length scale L is chosen to be of typical vehicle dimensions, a cellular automaton naturally arises.

In contrast, the moments $\int dv\, v^{\nu} f$ give the averaged quantities a macroscopic (or hydrodynamic) theory is built upon. For instance, $\nu=0$ is associated with the concentration $c(x,t)$, $\nu=1$ with the mean velocity $\overline{v(x,t)}$, $\nu=2$ with the velocity variance, and so forth. Although the infinite series $\nu\rightarrow\infty$ could, in theory, provide an exact description of the problem, practical calculations will break off at some finite ν^*, thus compromising the accuracy. The significance of the distribution function moments will, further below, be an argument for a series ansatz $f=\Sigma u_n(v)A_n(x,t)$.

II. DISTRIBUTION FUNCTION PROPERTIES

Let us first define the understanding and treatment of the distribution function f throughout this paper. The traffic situation under investigation shall be the evolution of some initial distribution propagating in one direction along a homogeneous stretch of road, i.e., no crossroads, oncoming traffic, or other obstructions shall be considered. Following Prigogine and Herman [8], the starting point for the analysis of the distribution function f for such a problem shall be the equation

$$\frac{\partial f}{\partial t}+v\,\frac{\partial f}{\partial x}=-\,\frac{f-f_0}{\tau}. \tag{1}$$

Note that the distribution of desired speeds f_0, as well as the time scale τ appearing in Eq. (1), is a macroscopic notion such that averaging over v gives the Navier-Stokes equation for traffic:

$$\left[\frac{\partial}{\partial t}+\overline{v}\frac{\partial}{\partial x}\right]\overline{v}=-\frac{\overline{v}-\overline{v^0}}{\tau}-\left[\frac{1}{c}\,\frac{\partial c}{\partial x}+\frac{\partial}{\partial x}\right](\overline{v^2}-\overline{v}^2). \tag{2}$$

As already argued in the Introduction, Eq. (2) is the second of a chain of equations for the moments of f:

$$\begin{aligned} c(x,t) &= \int dv\, f(x,v,t), \\ \overline{v(x,t,)} &= \int dv\, v\, f(x,v,t), \\ \overline{v^2} - \overline{v}^2 &= \int dv (v-\overline{v})^2 f(x,v,t), \\ &\vdots \end{aligned} \tag{3}$$

The equation corresponding to c is the continuity equation

$$\frac{\partial c}{\partial t} + \frac{\partial (c\overline{v})}{\partial x} = 0. \tag{4}$$

All hydrodynamic models [9,1,2] derive via inclusion of certain interaction terms and more or less restrictive assumptions from (2) and (4). Although, consequently, Eq. (1) has thus to be regarded as a macroscopic equation, it is potentially richer than equations of the same type as (2) and (4) because certain microscopic features can still be included, as shall be seen further below.

Let us now discuss the ingredients of Eq. (1). The macroscopic time scale τ governs the relaxation process from a given f onto f_0. In a microscopical picture, τ would result from averaging over all partial processes such as braking, accelerating, reaction on fluctuations, and delay times. Hence a simple relaxation form as in Eq. (1) has to be thought of as a rather crude macroscopic limit. It is possible to devise more sophisticated relaxation laws (one possibility will be given in Sec. III A), but the general assumption, namely, that global time scales for the evolution of the distribution function as a whole exist, cannot be circumvented and is, within this model, thought to cover all essential effects in traffic flow phenomena.

Equation (1) further assumes that there is a ''program'' drivers wish to follow, a distribution of desired speeds f_0. The form of Eq. (1) would imply that f_0 is the equilibrium distribution. This interpretation, though perfectly valid in the gas kinetic analog of (1), is, however, rather problematic in the context of traffic flow. For instance,

$$f_{\rm stat}(x,v,t) = f_{\rm stat}(v,c) \quad \text{with} \quad c = \text{const} \tag{5}$$

is a stationary solution of (1). Given an initial concentration different from zero only in some finite stretch of road, it is clear immediately that the homogeneous limit (5) will not be reached in the upstream regions of the initial cluster. This leads back to one of the principal problems of a physical traffic flow theory: Due to the absence of genuine molecular chaos, the existence and the definition of an equilibrium state of traffic are unclear (apart, of course, from the trivial equilibrium of standing traffic). It follows from this consideration that Eq. (1) can be regarded as useful only in a temporal regime where fluctuations are absorbed in a local equilibrium form $f_0 = f_0(v, c(x,t))$. Solutions near $c = \text{const}$ on the entire lane considered will have to be regarded as unphysical. Inherent in Eq. (1) is, furthermore, an ambiguity in the understanding of f_0, which will be discussed in the following section.

III. DESIRED SPEED DISTRIBUTION AND FUNDAMENTAL DIAGRAM

A. Ambiguity of desired speed distribution

The distribution of desired velocities can be understood in either of the following ways.

First, f_0 gives the distributions drivers would assume in a noninteracting traffic situation, i.e., in dilute traffic. Then only car characteristics and driver preferences would make up the statistics of f_0. This approach has been adopted by Prigogine and Herman [8], who therefore needed to include an additional interaction term on the right-hand side of the evolution equation (1):

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\frac{f-f_0}{\tau} + (1-P)c(\overline{v}-v)f, \tag{6}$$

P being the probability of passing. The stationary solutions (5) in the reasoning of Prigogine and Herman then arise from this interaction in the homogeneous limit:

$$\begin{aligned} f_{\rm stat} &= f_{\rm stat}(f_0^{\rm stat}(v), c), \\ f_{\rm stat} &= \frac{f_0^{\rm stat}}{1 - c\tau(1-P)(\overline{v}-v)}. \end{aligned} \tag{7}$$

The interaction term itself is found in analogy to the Boltzmann collision term in gas kinetics. It should be noted that this approach, starting from noninteracting vehicles and some form of interaction, is genuinely *ab initio*.

Second, f_0 is taken in such a form that the density dependence as in (7) is already acknowledged:

$$f_0 = f_0(v,c). \tag{8}$$

The form (8) takes the interactions implicitly into account. This leaves Eq. (1) in its simple form. Instead of the interaction term, however, some other sort of input is needed to determine f_0. The appropriate choice is the center piece of traffic engineering, the fundamental diagram $\overline{v} = V(c)$, relating the mean velocity to the traffic density. This relation itself has been the subject of a substantial body of work (see, for instance, [10]). In this paper it shall be assumed to exist and have the form of the fit given by Kerner and Kohnhäuser [1]:

$$V(c; v_0, c_{\max}) = v_0\left(\left\{1 + \exp\left[\left(\frac{c}{c_{\max}} - 0.25\right) \Big/ 0.06\right]\right\}^{-1} - 3.72\times 10^{-6}\right). \tag{9}$$

Although this second definition of the desired speed distribution is an *empirical* one and thus methodically inferior to that of Prigogine and Herman, the conserved simple form of (1) and the accurate fit (9) make it more suitable for analysis.

Both approaches have, of course, to be equivalent in the ensuing physics. This is achieved by identifying $\overline{v}$ in the first moment of (7)

$$\overline{v^{(0)}}=\overline{v}+\tau(1-P)c(\overline{v^2}-\overline{v}^2), \tag{10}$$

with the $V(c)$ of (9).

Let us define, then, that throughout this paper f_0 shall be understood to be of the form (8). The method to employ a simple theory with complicated interaction effects taken into account with some appropriately defined effective quantities has been proven to be a successful first approximation in many areas of physics such as, e.g., effective ionization energies in plasma physics and effective radii of interaction in nuclear physics.

B. Series representation of f_0

With the ansatz

$$f_0(v,c)=\sum_{n=1}^{\infty} u_n(v)A_n^{(0)}(c),$$

$$u_n=\sqrt{\frac{2}{\pi}}\sin\theta, \tag{11}$$

$$\theta=\frac{2v-v_{\max}-v_{\min}}{v_{\max}-v_{\min}},$$

it is now straightforward to find f_0 via its moments. We stress again that, in the above-defined picture, its principal feature is the incorporation of the fundamental diagram (9) that will enter in the first moment. The zeroth moment is given by the density and the higher ones determine the statistical properties. With the physical request that $\overline{v}\pm\sqrt{(\overline{v^2}-\overline{v}^2)}$ be positive, one finds, up to second order,

$$A_1^{(0)}=\sqrt{\frac{2}{\pi}}\frac{2c}{v_{\max}-v_{\min}},$$

$$A_2^{(0)}=\sqrt{\frac{2}{\pi}}\frac{8c}{(v_{\max}-v_{\min})^2}\left(V(c)-\frac{v_{\max}+v_{\min}}{2}\right),$$

$$A_3^{(0)}=\sqrt{\frac{2}{\pi}}\frac{32c}{(v_{\max}-v_{\min})^3}\times\{V(c)[v_{\max}+v_{\min}-V(c)]+h(c)\}, \tag{12}$$

$$h(c)=h_0\left(\frac{V(c)}{V(0)}\right)^{\gamma}-\frac{v_{\max}^{\gamma}V(c)[v_{\max}-V(c)]}{V(c)+v_{\max}^{\gamma}},$$

$$h_0=(\overline{v^2}-V^2)|_{c=0}-V(0)[v_{\max}-V(0)]\left[1-\frac{v_{\max}^{\gamma}}{V(0)+v_{\max}^{\gamma}}\right],$$

$$A_4^{(0)}=0.$$

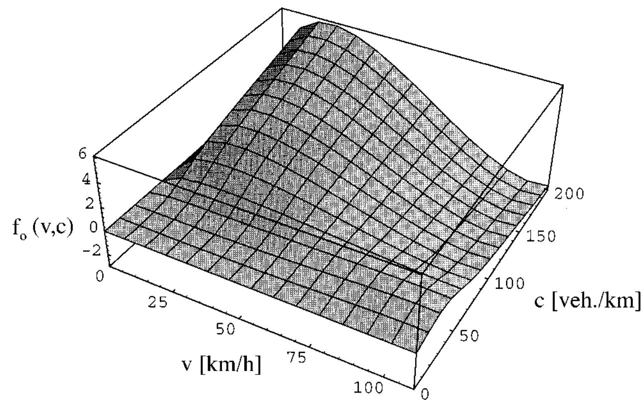

FIG. 1. Equilibrium distribution function $f_0(v,c)$.

In (12), γ is a numerical parameter and $h(c)$ an auxiliary function to ensure the above-mentioned property of the velocity variance and smooth numerical behavior. Figure 1 shows the f_0 according to (12). The nonsensical negative values are due to the relatively early breakoff of the series (11). This drawback can be easily corrected in more ambitious actual simulations. Since the purpose of the present work is rather to highlight the theoretical aspects of an f-based traffic flow theory, the given form of f_0 may be sufficient. We stress that the characteristics of the fundamental diagram—higher traffic density forces the velocity expectation value to decrease—are clearly apparent in Fig. 1.

Figure 2 shows the fundamental diagram according to (9) with the corresponding variance. The parameter $\gamma=2$ has been chosen to have monotonic behavior of the variance and $(\overline{v^2}-\overline{v}^2)|_{c=0}=20.467$ km/h for the variance of the noninteracting traffic.

IV. MODE SOLUTION FOR THE DISTRIBUTION FUNCTION

A. Analytical results

The representation (11) and (12) of the stationary, or ''equilibrium,'' distribution of desired speeds in the defined understanding suggests a similar treatment of the full time- and space-dependent equation (1). In fact, the definitions of (11) can be fully adopted once the substitution

$$A_n^{(0)}(c)\to A_n(c(x,t)) \tag{13}$$

is made. The above form clearly displays the local equilibrium character of the solution.

Equation (1) itself transforms into

$$\left[\frac{\partial}{\partial t}+\frac{v_{\max}+v_{\min}}{2}\frac{\partial}{\partial x}\right]A_n+\frac{v_{\max}-v_{\min}}{4}\frac{\partial}{\partial x}[A_{n-1}+A_{n+1}]+\frac{A_n-A_n^{(0)}(c_{\mathrm{ref}}(x,t))}{\tau}=0, \tag{14}$$

where $n\geqslant 1$, $A_n^{(0)}$ is the corresponding mode of f_0, and $A_0=0$ per definition.

Equation (14) demands several comments. First, it has to be pointed out that the nonlinearity problem in (1) has been transformed into simple mode coupling within an infinite

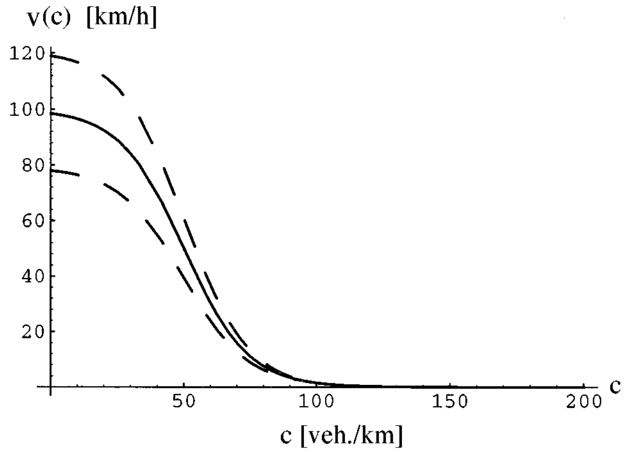

FIG. 2. Fundamental diagram $V(c)$ with variance, $v_0=100$ km/h and $k_{\max}=200$ vehicles/km.

TABLE I. Values of parameters used in various figures.

Parameter	Units	Figures
homogeneous background concentration $c_{\rm bg}$	10 vehicles/km	3–6
	90 vehicles/km	7 and 8
location of initial velocity distribution maximum	120 m	3, 4, 7, and 8
	80 m	5 and 6
location of initial variance distribution maximum	110 m	3, 4, 7, and 8
	90 m	5 and 6

chain of equations. Second, the relaxation term has been written in a form that allows one to stretch the concept of local equilibrium. In $A_n^{(0)}(c_{\rm ref}(x,t))$ a reference density $c_{\rm ref}$ has been introduced in order to take into account some peculiarities of the traffic situation:

$$c_{\rm ref}(x,t,)=\frac{1}{L}\int_x^{x+L}dxc(x,t). \tag{15}$$

In the ansatz (15) it has been acknowledged that the individual driver will react to a certain traffic density not only at his immediate position x, but within a certain neighborhood, characterized by the typical length L. Furthermore, the driver will be influenced only by the state of traffic in front of it, thus the averaging over the interval $[x,x+L]$. The eventual stationary solution $f_{\rm stat}(v,c=\text{const})$ and the conservation of the number of vehicles will not be affected by the prescription (15). In the interesting temporal regime, and in marked difference to the hydrodynamical models, it can, however, be expected that the microscopic features of traffic flow taken into account via (15) improve the consistency of the model. It is this kind of pseudomicroscopic theory buildup that seems to support the distribution function f as the central entity of traffic flow modeling. With the strict local equilibrium from macroscopic theory one would have the constraint

$$\int dvA_1^{(0)}(v,c(x,t))=\int dvA_1(x,t) \tag{16}$$

in order to reproduce the density $c(x,t)$. This would lead to a different equation for A_1 due to the absence of τ. The actual processes of driving embedded in a stream of cars seems to be better modeled, though, with the more flexible reference density $c_{\rm ref}$.

By a similar reasoning, Eq. (14) allows for individual relaxation times τ_n for the different modes A_n. Such a model refinement would correspond to grouping the relaxation processes with respect to their significance for the respective modes. Although such a procedure cannot recover the true microscopic processes, it is certainly closer to reality than a global time scale τ. (For simplicity reasons, however, this paper shall be restricted to just one time scale.)

Analysis of (14) is facilitated by a Fourier-Laplace transformation with respect to time and space:

$$\widehat{\widetilde{A}}_m=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dxe^{-ikx}\int_0^{+\infty}dte^{-st}A_m(x,t). \tag{17}$$

Then Eq. (14) transforms into

$$\widehat{\widetilde{A}}_n\left[\tau^{-1}+s+\frac{ik}{2}(v_{\max}+v_{\min})\right]$$
$$=\widetilde{B}_n-\frac{ik}{4}(v_{\max}-v_{\min})[\widehat{\widetilde{A}}_{n-1}+\widehat{\widetilde{A}}_{n+1}]+\frac{\widehat{\widetilde{A}}_n^{(0)}}{\tau}. \tag{18}$$

Here the $\widetilde{B}_n$ denotes the spatial Fourier transform of a given initial mode $B_n(x)=A_n(x,t=0)$.

It is now possible to obtain a formally exact solution of Eq. (18) (for details see the Appendix)

$$\begin{aligned}A_n(x,t)=e^{-t/\tau}\Bigg\{&B_n\left(x-\frac{t}{2}(v_{\max}+v_{\min})\right)-\frac{v_{\max}-v_{\min}}{4}\\&\times\int_0^t dze^{z/\tau}\frac{\partial}{\partial x}[A_{n-1}+A_{n+1}]_{x-(t-z)/2(v_{\max}+v_{\min})}\\&+\frac{1}{\tau}\int_0^t dze^{z/\tau}A_n^{(0)}\left(x-\frac{t}{2}(v_{\max}+v_{\min})\right)\Bigg\}.\end{aligned} \tag{19}$$

The stationary limit $t\rightarrow\infty$ is immediately apparent in (19): The initial distribution $\{B_n\}$ dies off exponentially, the spatial derivatives vanish, and the solution is eventually given by $A_n^{(0)}(v,c(x,t)\rightarrow\text{const})$. The complexity of the solution is largely given by the mode coupling term and the response integrals in the second and third terms on the right-hand side of (19).

The practical value of this exact solution is limited due to the interdependence of the modes that reflects the original nonlinearity of the evolution equation. If, however, at some time t^* the state of the system is given, a Taylor expansion of (19) provides an infinitesimal evolution operator. Within first order in a time increment Δ we have

$$
\begin{aligned}
A_n(x,t^*+\Delta) = {} & e^{-(t^*+\delta)/\tau} B_n\left(x-\frac{t^*+\Delta}{2}(v_{\max}+v_{\min})\right) + e^{-t^*/\tau}\left(1-\frac{\Delta}{\tau}\right)\int_0^{t^*} dz \\
& \times e^{z/\tau}\left[-\frac{v_{\max}+v_{\min}}{4}\partial_x(A_{n-1}+A_{n+1})_{x-(t^*-z)/2(v_{\max}+v_{\min})} + \frac{1}{\tau}A_n^{(0)}\left(x-\frac{t^*+\Delta}{2}(v_{\max}+v_{\min})\right)\right] \\
& + \Delta[\partial_x(A_{n-1}+A_{n+1})_x + A_n^{(0)}(x)] + \Delta e^{-t^*/\tau}\int_0^{t^*} dz e^{z/\tau}\left[\frac{v_{\max}^2-v_{\min}^2}{4}\partial_x^2(A_{n-1}+A_{n+1})_{x-(t^*-z)/2(v_{\max}+v_{\min})}\right. \\
& \left. -\frac{v_{\max}+v_{\min}}{2\tau}A_n^{(0)}\left(x-\frac{t^*+\Delta}{2}(v_{\max}+v_{\min})\right)\right]. \qquad (20)
\end{aligned}
$$

With (20) as the computational prescription, it is possible to perform numerical simulations of the evolution of certain traffic situations. In the following a number of characteristic cases are shown.

B. Numerical results

The test cases of calculations on the basis of (20) in this section have been chosen so as to highlight typical problems of traffic evolution. Basis of the computations are the first three moments A_n, $n=1,2,3$ equivalent to the treatment of f_0 in Sec. II A. It is clear that this level of accuracy will not be sufficient for realistic simulation; numerical errors, especially oscillations, will have to be expected. Nevertheless, these examples clearly show that the formalism presented is able to reproduce a number of characteristic features. The simulation shows the evolution of an initial population over $t=4$ sec on a stretch of road of length $x=500$ m. Because of the mode decomposition it is particularly easy to identify effects of inhomogeneities in concentration, velocity, and variance. Their initial distributions have been chosen to be Gaussian shaped with the same width and variable location as that of the maximum.

The following parameters have been left constant throughout all runs: velocity range, $v_{\min}=0 \leqslant v \leqslant v_{\max}=35$ m/sec; free traffic speed in f_0, $v_0=27.78$ m/sec; saturation vehicle concentration, $c_{\max}=200$ vehicles/km; numerical fit parameter in f_0, $\gamma=2$; forward averaging interval according to (15), $L=20$ m; a Gaussian-shaped initial concentration distribution with maximum $c(x=100m,\ t=0)=100$ vehicles/km; a Gaussian-shaped initial velocity distribution with maximum $v=27.78$ m/sec; a Gaussian-shaped initial velocity variance distribution with maximum $\mathrm{var}_v=209.16$ $\mathrm{m}^2/\mathrm{sec}^2$; and relaxation time $\tau=5$ sec. The values of the varied parameters in the respective figures is given in Table I.

Figures 3 and 4 show the evolution of the concentration and velocity of an initial distribution where the fastest cars are 20 m ahead of the highest density and the background concentration is low. The result for the concentration (Fig. 3) shows a decomposition: a gap arises between bulk and leaders. Furthermore, the initial concentration peak flattens out; after 4 sec the maximum density has fallen from 100 vehicles/km to just about approximately 50 vehicles/km. In the corresponding velocities (Fig. 4) it can clearly be seen that areas with high traffic density have low mean speed and vice versa. The simulation also correctly reproduces an increasing velocity of the dilute background: these cars try to assume their equilibrium speed $V(c_{\mathrm{bg}})$. The cars on the slow shoulder of the Gaussian hardly move at all.

In Figs. 5 and 6 the asymmetry of the velocity and variance has been mirrored: the fastest car and the variance maximum are now upstream of the highest density. This results in a temporarily increasing density when the faster cars overtake the bulk. After this the same clustering as in Fig. 3 occurs where the main peak retains a slightly higher value due to the cars still overtaking from behind. To the process of overtaking corresponds a fall in the mean speed, as can be seen in Fig. 6, since the faster cars are obstructed by dense traffic. In the contrasting case (Fig. 4), the fastest cars have diminishing traffic density in front of them and are thus less affected by interaction. It has to be cautioned, though, that the maximum appearing for higher times in Fig. 6 is a numerical artifact.

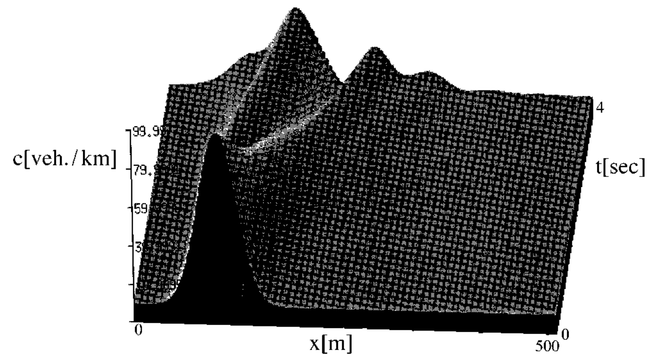

FIG. 3. Low background density, velocity, and variance maxima downstream.

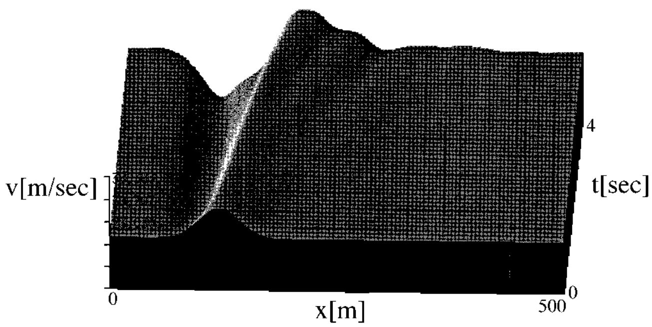

FIG. 4. Low background density, velocity, and variance maxima downstream.

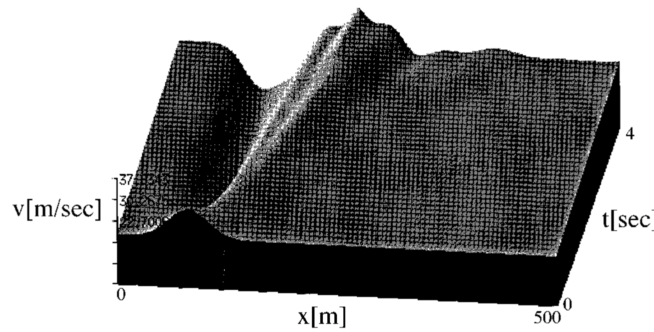

FIG. 6. Low background density, velocity, and variance maxima upstream.

In Figs. 7 and 8 the locations of the concentration, velocity, and variance distributions are the same as in Figs. 3 and 4; the concentration peak value has been left at 100 vehicles/km, but the background density has been set at $c_{\rm bg}=90$ vehicles/km. In this case the small density fluctuation reveals a supercritical condition and leads to a jam, i.e., a density increase. The same behavior has been observed in [1]. In comparison with Fig. 3, it can be stated that the same fluctuation amplitude can lead to completely different states of traffic, indicating a phase transition. Such a transition, being of nonequilibrium nature, is not easily defined. The shape of the fluctuations as well as the multitude of environmental parameters will influence the outcome, such that the critical background density $c_{\rm bg}^{\rm crit}$ that separates the relaxation and congestion regimes, will be a complicated functional

$$c_{\rm bg}^{\rm crit}=c_{\rm bg}^{\rm crit}[\{A_n\};\mathcal{P}].$$

where $\mathcal{P}$ denotes parameters. For the time being we restrict ourselves to the statement that the simple analysis carried out above is capable of describing this criticality. Figure 8, contrary to Fig. 4, also clearly shows the decreasing mean speed of the dense background traffic according to the fundamental diagram.

Other tests have been carried out, but were not shown here in order to maintain clarity of the representation. For instance, the relaxation time τ has been varied and, as expected, a slower decrease of the concentration peaks with otherwise topologically equivalent behavior for higher τ was established. In order to restrict the parameter variations to relevant cases, more work needs to be carried out in the diagnostics of traffic phenomena. New measurements [11,12] may provide a useful basis for such future investigations.

CONCLUSION

The aim of this paper was to revive the distribution function $f(x,v,t)$ as the center piece of a traffic flow model. The basis of the analysis is the (formally) interaction-free macroscopic equation (1), which thus facilitates a simple analysis. The interactions have been incorporated via the so-called fundamental diagram in the ''equilibrium'' distribution f_0. In contrast to the approach from first principles of Prigogine and Herman, our theory has to be labeled empirical since the interactions are described by the measured relation $\overline{v}=V(c)$. Justification for such an approach lies in the ill-defined nature of microscopical interaction, the credibility improvement if $V(c)$ is acknowledged, and the analytical progress that can be made. By means of a mode decoupling, i.e., a series ansatz for the distribution function $f(x,v,t)$, the moments of f and their behavior can be easily identified. A rather crude numerical evaluation already reproduces typical traffic features. Of special importance is the observed criticality with respect to fluctuations on a homogeneous background density. Future work will have to dwell on this phenomenon.

Although the numerical studies presented have been restricted to a rather limited problem, it has to be stressed that the distribution function f can provide a route to real-world simulations. Crossroads, traffic lights, and similar topologi-

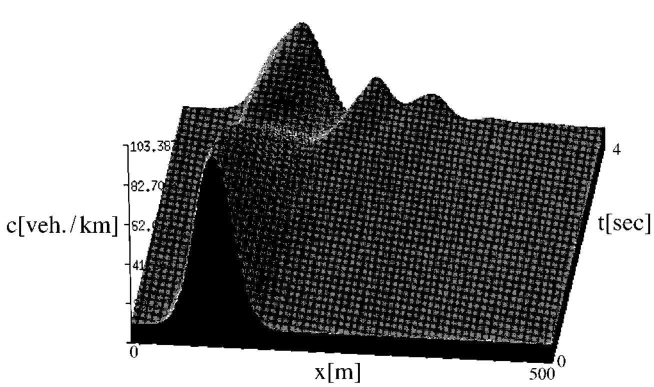

FIG. 5. Low background density, velocity, and variance maxima upstream.

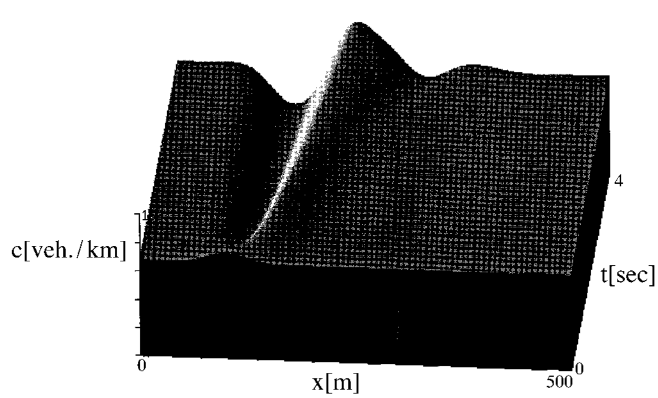

FIG. 7. High background density, velocity, and variance maxima downstream.

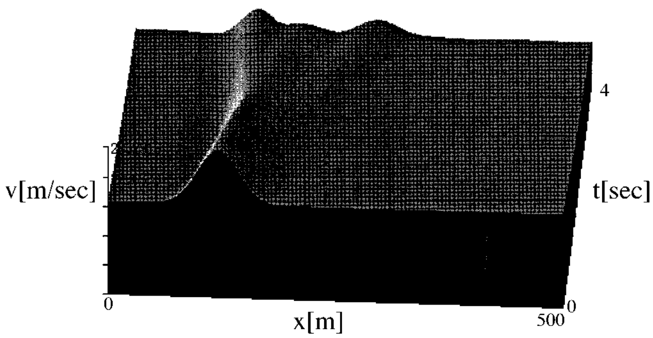

FIG. 8. High background density, velocity, and variance maxima downstream.

cal singularities just have to be translated into prescriptions of how to alter the number of vehicles at time t and space point x. These rules combined with the flow propagation according to the presented analysis will lead to realistic models. As commented before, the use of the distribution function f and its mode decomposition allows for pseudomicroscopic model refinements such as different relaxation times or effective interaction radii. True microscopic behavior cannot, of course, be described with an equation such as (1). Since, however, the microscopic processes of vehicle driving seem to lie beyond the realm of physics, an argument arises whose approach, on the basis of averaged quantities such as, e.g., the fundamental diagram $V(c)$, may be more reliable.

Finally, the problems of traffic flow modeling should be put in perspective since the harder problems of modeling seem to concern topological issues (crossroads, path choice, etc.) such as, for instance, those addressed in [13,14]. It may be summarized that realistic models of vehicular traffic with prognostic power still pose a great problem that can, within certain limits, be tackled with the use of theoretical physics.

APPENDIX

Starting with Eq. (18), the solution in Fourier-Laplace space is obtained immediately:

$$\widehat{\widetilde{A}}_n = \frac{\widetilde{B}_n - \frac{ik}{4}(v_{\max}-v_{\min})[\widehat{\widetilde{A}}_{n-1}+\widehat{\widetilde{A}}_{n+1}] + \tau^{-1}\widehat{\widetilde{A}}_m^{(0)}}{s+\tau^{-1}+\frac{ik}{2}(v_{\max}+v_{\min})} = T_1+T_2+T_3. \quad \text{(A1)}$$

Note that the $\widetilde{B}_n$ arise from

$$\widehat{\widetilde{\partial_t A_n}}(k,s) = -\widetilde{A}_n(k,t=0) + s\widehat{\widetilde{A}}_n(k,s).$$

The first term on the right-hand side, T_1, is independent of time t and its Laplace conjugate s transforms as a simple translation

$$\mathcal{F}^{-1}\mathcal{L}^{-1}[T_1] = e^{-t/\tau}B_n\left(x-\frac{t}{2}(v_{\max}+v_{\min})\right). \quad \text{(A2)}$$

The inverse Laplace transformation of the second and third terms on the right-hand side of Eq. (A1), T_2 and T_3, involves a convolution, but is otherwise straightforward. With

$$\widehat{\widetilde{\partial_x A_n}} = ik\widehat{\widetilde{A}}_n,$$

one finds

$$\mathcal{F}^{-1}\mathcal{L}^{-1}[T_2] = -\frac{v_{\max}-v_{\min}}{4}\int_0^t dz e^{-(t-z)/\tau} \times \partial_x[A_{n-1}+A_{n+1}]_{x-(t-z)/\tau(v_{\max}+v_{\min})},$$

$$\mathcal{F}^{-1}\mathcal{L}^{-1}[T_3] = \int_0^t dz\frac{1}{\tau}e^{-(t-z)/\tau}A_n^{(0)}\left(x-\frac{t}{2}(v_{\max}+v_{\min})\right). \quad \text{(A3)}$$

Putting together (A2) and (A3), one arrives at the final result (19).

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